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On generalized Stieltjes functions

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Abstract

It is shown that a function f is a generalized Stieltjes function of order $\lambda > 0$ if and only if $x^{1-\lambda}(x^{\lambda-1+k}f(x))^{(k)}$ is completely monotonic for all $k \geq 0$, thereby complementing a result due to Sokal. Furthermore, a characterization of those completely monotonic functions f for which $x^{1-\lambda}(x^{\lambda-1+k}f(x))^{(k)}$ is completely monotonic for all $k \leq n$ is obtained in terms of properties of the representing measure of f .

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1 Introduction

In this paper we investigate a real-variable characterization of generalized Stieltjes functions obtained by Sokal, see [9].

Let $\lambda > 0$ be given. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called a generalized Stieltjes function of order λ if

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} + c,$$

where μ is a positive measure on $[0, \infty)$ making the integral converge for $x > 0$ and $c \geq 0$.

The class of ordinary Stieltjes functions is the class of generalized Stieltjes functions of order 1.

A C^∞ -function f on $(0, \infty)$ is completely monotonic if $(-1)^n f^{(n)}(x) \geq 0$ for all $n \geq 0$ and all $x > 0$. Bernstein's theorem characterizes these functions

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as Laplace transforms of positive measures: f is completely monotonic if and only if there exists a positive measure μ on $[0, \infty)$ such that $t \mapsto e^{-xt}$ is integrable w.r.t. μ for all $x > 0$ and

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

cf. [11, p. 161]. We remark that f is a generalized Stieltjes function of order λ if and only if

$$f(x) = \int_0^\infty e^{-xt} t^{\lambda-1} \varphi(t) dt + c, \quad x > 0 \quad (1)$$

for some completely monotonic function φ , and some non-negative number c . See [4, Lemma 2.1].

Sokal (see [9]) introduced for $\lambda > 0$ the operators

$$T_{n,k}^\lambda(f)(x) \equiv (-1)^n x^{-(n+\lambda-1)} \left(x^{k+n+\lambda-1} f^{(n)}(x) \right)^{(k)}, \quad n, k \geq 0$$

and obtained the following characterization.

Theorem 1.1 *The following are equivalent for a C^∞ -function f defined on $(0, \infty)$.*

- (a) f is a generalized Stieltjes function of order λ ;
- (b) $T_{n,k}^\lambda(f)(x) \geq 0$ for all $x > 0$, and $n, k \geq 0$.

Sokal's characterization is an extension of Widder's characterization of the class of ordinary Stieltjes functions: f is a Stieltjes function if and only if the function $(x^k f(x))^{(k)}$ is completely monotonic for all $k \geq 0$. (See [10].)

In [5, Theorem 1.5] an analogue of Sokal's result where the function φ in (1) is absolutely monotonic is obtained. See also [3, Theorem 2] for a result complementing [5, Theorem 1.1].

Remark 1.2 *Notice that, by Leibniz' rule,*

$$x^{-(n+\lambda-1)} \left(x^{k+n+\lambda-1} f^{(n)}(x) \right)^{(k)} = \sum_{j=0}^k \binom{k}{j} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+j+\lambda)} x^j f^{(n+j)}(x).$$

In this paper we first show that condition (b) in Sokal's theorem above can be replaced by the condition that

$$c_k^\lambda(f)(x) \equiv x^{1-\lambda} (x^{\lambda-1+k} f(x))^{(k)}$$

is completely monotonic for all k . There is a simple relation between $T_{n,k}^\lambda(f)$ and $c_k^\lambda(f)$:

Proposition 1.3 *The relation*

$$T_{n,k}^\lambda(f)(x) = (-1)^n c_k^\lambda(f)^{(n)}(x)$$

holds for any $n, k \geq 0$ and $x > 0$.

Corollary 1.4 *The following are equivalent for a function $f \in C^\infty((0, \infty))$.*

- (i) *f is a generalized Stieltjes function of order λ ;*
- (ii) *$c_k^\lambda(f)$ is completely monotonic for all $k \geq 0$.*
- (iii) *$T_{n,k}^\lambda(f) \geq 0$ for all $n \geq 0$ and all $k \geq 0$.*

In [4] the generalized Stieltjes functions corresponding to measures having moments of all orders were characterized in terms of properties of remainders in asymptotic expansions. (A measure μ has moments of all orders if any polynomial is integrable w.r.t. μ .) In view of the results in the present paper we notice the following corollary. The proof follows by combining Corollary 1.4 with [4, Theorem 3.2] and [4, Lemma 3.1].

Corollary 1.5 *The following are equivalent for a function $f : (0, \infty) \rightarrow \mathbb{R}$.*

- (i) *f is a generalized Stieltjes function corresponding to a measure μ having moments of all orders;*
- (ii) *$c_k^\lambda(f)$ is completely monotonic for all $k \geq 0$ and the function $x^{\lambda-1}f(x)$ admits for any n an asymptotic expansion*

$$x^{\lambda-1}f(x) = \sum_{k=0}^{n-1} \frac{\alpha_k}{x^{k+1}} + r_n(x),$$

in which $x^n r_n(x) \rightarrow 0$ as $x \rightarrow \infty$.

In the affirmative case, $\alpha_k = (-1)^k(\lambda)_k s_k(\mu)/k!$ where $s_k(\mu)$ is the k 'th moment of μ , and r_n has the representation

$$r_n(x) = \frac{(-1)^n x^{\lambda-1}}{\Gamma(\lambda)} \int_0^\infty e^{-xt} t^{\lambda-1} \xi_n(t) dt,$$

where ξ_n belongs to $C^\infty([0, \infty))$, and satisfies $\xi_n^{(j)}(0) = 0$ for $j \leq n-1$ and $0 \leq \xi_n^{(n)}(t) \leq s_n(\mu)$ for $t \geq 0$. Furthermore,

$$c_n^\lambda(f)(x) = x^{1-\lambda} (x^n r_n(x))^{(n)} = c_n^\lambda \left(\mathcal{L} \left(\frac{t^{\lambda-1} (-1)^n \xi_n(t)}{\Gamma(\lambda)} \right) \right) (x).$$

Our aim is also to characterize, for any given positive integer N , those functions f for which $c_0^\lambda(f), \dots, c_N^\lambda(f)$ are completely monotonic. In the case where $\lambda = 1$ this has been carried out in [6], but the case of general λ requires, as we shall see, additional insight.

We thus introduce the classes \mathcal{C}_N^λ as

$$\mathcal{C}_N^\lambda = \{f \in C^\infty((0, \infty)) \mid c_k^\lambda(f) \text{ is completely monotonic for } k = 0, \dots, N\}.$$

We shall use some distribution theory so we briefly describe our notation. The action of a distribution u on a test function φ (an infinitely often differentiable function of compact support in $(0, \infty)$) is denoted by $\langle u, \varphi \rangle$. The distribution ∂u is defined via $\langle \partial u, \varphi \rangle = -\langle u, \varphi' \rangle$. A standard reference to distribution theory is [7].

Our results can be formulated as follows.

Theorem 1.6 *Let $\lambda > 0$ be given, and let $N \geq 1$. The following properties of a function $f : (0, \infty) \rightarrow \mathbb{R}$ are equivalent.*

- (a) $f \in \mathcal{C}_N^\lambda$;
- (b) f can be represented as

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1} d\mu(s),$$

where $c \geq 0$, and μ is a positive measure on $(0, \infty)$ for which $\mu_k \equiv (-1)^k s^k \partial^k \mu$, (in distributional sense) is a positive measure such that

$$\int_0^\infty e^{-xs} s^{\lambda-1} d\mu_k(s) < \infty, \quad k = 0, \dots, N.$$

In the affirmative case,

$$c_k^\lambda(f)(x) = x^{1-\lambda} \left(x^{\lambda-1+k} f(x) \right)^{(k)} = \int_0^\infty e^{-xs} s^{\lambda-1} d\mu_k(s) + (\lambda)_k c$$

for $k = 0, \dots, N$.

We notice the following corollary characterizing those non-negative functions f for which $c_1^\lambda(f)$ is completely monotonic. The proof follows from Proposition 2.3 and Lemma 3.5.

Corollary 1.7 *Let f be a non-negative C^∞ -function defined on $(0, \infty)$. Then $x^{1-\lambda} (x^\lambda f(x))' = \lambda f(x) + x f'(x)$ is completely monotonic if and only if*

$$f(x) = \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty e^{-xs} s^{\lambda-1} \int_s^\infty \frac{d\mu(t)}{t^\lambda} ds,$$

for some non-negative numbers α and β and some positive measure μ on $(0, \infty)$ making the integral convergent.

Remark 1.8 *It is easy to see that $e^{-xs} s^{\lambda-1} \int_s^\infty d\mu(t)/t^\lambda$ is integrable on $(0, \infty)$ if and only if $s^{\lambda-1} \int_s^\infty d\mu(t)/t^\lambda$ is integrable at 0, and that this is the case if and only if $\int_0^1 d\mu(t) < \infty$ and $\int_1^\infty d\mu(t)/t^\lambda < \infty$.*

Corollary 1.7 can be reformulated as follows. Let g be a non-negative C^∞ -function on $(0, \infty)$. Then $x^{1-\lambda} g'(x)$ is completely monotonic if and only if

$$g(x) = \alpha x^\lambda + \beta + \int_0^\infty \int_0^{xt} e^{-u} u^{\lambda-1} du \frac{d\mu(t)}{t^\lambda}. \quad (2)$$

Formulated in this way the corollary is related to the class of Bernstein functions. A Bernstein function is by definition a non-negative function g on $(0, \infty)$ for which g' is completely monotonic. These functions admit an integral representation (see [8, Theorem 3.2] or [2]), which we for the reader's convenience state here: g is a Bernstein function if and only if

$$g(x) = \alpha x + \beta + \int_0^\infty (1 - e^{-xt}) d\nu(t),$$

where α and β are non-negative numbers, and ν , called the Lévy measure, is a positive measure on $(0, \infty)$ satisfying $\int_0^1 t d\nu(t) < \infty$ and $\int_1^\infty d\nu(t) < \infty$.

When $\lambda = 1$, we have

$$\int_0^{xt} e^{-u} u^{\lambda-1} du = 1 - e^{-xt},$$

and (2) reduces to the integral representation of a Bernstein function with the corresponding Lévy measure being $d\mu(t)/t$. Corollary 1.7 contains a characterization of what could be called “generalized Bernstein functions of order λ ”.

2 Proofs

Proof of Proposition 1.3: The key to the proof is the following relation

$$T_{n,k}^\lambda(f)(x) = (-1)^n \left(\sum_{j=0}^k (\lambda - 1)_{k-j} \binom{k}{j} (x^j f(x))^{(j)} \right)^{(n)}, \quad (3)$$

which we verify now. A standard application of Leibniz' formula yields

$$\begin{aligned} (x^j f(x))^{(j+n)} &= \sum_{l=0}^{n+j} \binom{n+j}{l} (x^j)^{(l)} f^{(n+j-l)}(x) \\ &= \sum_{l=0}^j \binom{n+j}{l} \frac{j!}{(j-l)!} x^{j-l} f^{(n+j-l)}(x) \\ &= \sum_{m=0}^j \binom{n+j}{j-m} \frac{j!}{m!} x^m f^{(n+m)}(x). \end{aligned}$$

Hence, the right hand side of (3) equals

$$\begin{aligned} &(-1)^n \sum_{j=0}^k (\lambda - 1)_{k-j} \binom{k}{j} (x^j f(x))^{(j+n)} \\ &= (-1)^n \sum_{j=0}^k (\lambda - 1)_{k-j} \binom{k}{j} \sum_{m=0}^j \binom{n+j}{j-m} \frac{j!}{m!} x^m f^{(n+m)}(x) \\ &= (-1)^n \sum_{m=0}^k \left\{ \sum_{j=m}^k (\lambda - 1)_{k-j} \binom{k}{j} \binom{n+j}{j-m} \frac{j!}{m!} \right\} x^m f^{(n+m)}(x). \end{aligned}$$

The expression in the brackets can be written in another form. Indeed

$$\sum_{j=m}^k (\lambda - 1)_{k-j} \binom{k}{j} \binom{n+j}{j-m} \frac{j!}{m!} = \binom{k}{m} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+m+\lambda)},$$

by a corollary to the Chu-Vandermonde identity (see [1, p. 70]). This gives us

$$\begin{aligned} &(-1)^n \left(\sum_{j=0}^k (\lambda - 1)_{k-j} \binom{k}{j} (x^j f(x))^{(j)} \right)^{(n)} \\ &= (-1)^n \sum_{m=0}^k \binom{k}{m} \frac{\Gamma(n+k+\lambda)}{\Gamma(n+m+\lambda)} x^m f^{(n+m)}(x) = T_{n,k}^\lambda(f)(x). \end{aligned}$$

For $n = 0$ the identity reads

$$\sum_{j=0}^k (\lambda - 1)_{k-j} \binom{k}{j} (x^j f(x))^{(j)} = T_{0,k}^\lambda(f)(x) = c_k^\lambda(f)(x),$$

and the proposition is proved. \square

To prove Theorem 1.6 we need a few preliminary results.

Lemma 2.1 *For $f \in C^\infty((0, \infty))$ we have*

$$c_{k+1}^\lambda(f)(x) = (\lambda + k)c_k^\lambda(f)(x) + xc_k^\lambda(f)'(x).$$

Proof. This follows by computation:

$$\begin{aligned} c_{k+1}^\lambda(f)(x) &= x^{1-\lambda}(xx^{\lambda-1+k}f(x))^{(k+1)} \\ &= x^{2-\lambda}(x^{\lambda-1+k}f(x))^{(k+1)} + (k+1)x^{1-\lambda}(x^{\lambda-1+k}f(x))^{(k)} \\ &= x^{2-\lambda}(x^{\lambda-1}c_k^\lambda(f)(x))' + (k+1)c_k^\lambda(f)(x) \\ &= xc_k^\lambda(f)'(x) + (\lambda + k)c_k^\lambda(f)(x). \end{aligned}$$

\square

Proposition 2.2 *Suppose that $f \in \mathcal{C}_N^\lambda$, and let for $k = 0, \dots, N$*

$$c_k^\lambda(f)(x) = \int_0^\infty e^{-xs} d\mu_k(s) + b_k,$$

where μ_k is a positive measure on $(0, \infty)$ and $b_k \geq 0$. Then, in the distributional sense,

$$(-1)^k s^k \partial^k (s^{1-\lambda} \mu_0) = s^{1-\lambda} \mu_k.$$

Proof. From Lemma 2.1 it follows that (for $k \leq N - 1$)

$$\begin{aligned} \int_0^\infty e^{-xs} d\mu_{k+1}(s) + b_{k+1} &= (\lambda + k) \int_0^\infty e^{-xs} d\mu_k(s) + (\lambda + k)b_k \\ &\quad - x \int_0^\infty s e^{-xs} d\mu_k(s). \end{aligned}$$

Letting $x \rightarrow \infty$ yields $b_{k+1} = (\lambda + k)b_k$ so that

$$\begin{aligned} \frac{1}{x} \int_0^\infty e^{-xs} d\mu_{k+1}(s) &= (\lambda + k) \frac{1}{x} \int_0^\infty e^{-xs} d\mu_k(s) \\ &\quad - \int_0^\infty s e^{-xs} d\mu_k(s). \end{aligned}$$

By the uniqueness of the Laplace transform we obtain

$$s\mu_k = ((\lambda + k)\mu_k - \mu_{k+1}) * m.$$

(Here, m denotes Lebesgue measure on $(0, \infty)$.) We get by differentiation (as distributions) that

$$s\partial\mu_k = (\lambda + k - 1)\mu_k - \mu_{k+1}.$$

We shall obtain the assertion in the proposition by induction, using this recursive relation: for $k = 0$ the assertion is valid. Before verifying the induction step notice that

$$s\partial(s^k\partial^k(s^{1-\lambda}\mu_0)) = ks^k\partial^k(s^{1-\lambda}\mu_0) + s^{k+1}\partial^{k+1}(s^{1-\lambda}\mu_0).$$

Suppose now that the assertion holds for k . Then

$$\begin{aligned} s^{k+1}\partial^{k+1}(s^{1-\lambda}\mu_0) &= s\partial(s^k\partial^k(s^{1-\lambda}\mu_0)) - ks^k\partial^k(s^{1-\lambda}\mu_0) \\ &= s\partial((-1)^k s^{1-\lambda}\mu_k) - k(-1)^k s^{1-\lambda}\mu_k \\ &= (-1)^k \{s(1-\lambda)s^{-\lambda}\mu_k + s^{1-\lambda}s\partial\mu_k - ks^{1-\lambda}\mu_k\} \\ &= (-1)^k s^{1-\lambda} \{(1-\lambda)\mu_k + (\lambda + k - 1)\mu_k - \mu_{k+1} - k\mu_k\} \\ &= (-1)^{k+1} s^{1-\lambda}\mu_{k+1}. \end{aligned}$$

The assertion holds also for $k + 1$, and the assertion follows. \square

Proof that (a) implies (b) in Theorem 1.6. If $f \in \mathcal{C}_N^\lambda$ then the function $c_k^\lambda(f)$ is completely monotonic for $0 \leq k \leq N$. In particular

$$f(x) = c_0^\lambda(f)(x) = \int_0^\infty e^{-xs} d\mu_0(s) + b_0 = \int_0^\infty e^{-xs} s^{\lambda-1} d(s^{1-\lambda}\mu_0)(s) + b_0.$$

Let $\mu = s^{1-\lambda}\mu_0$ and notice that by Proposition 2.2 $(-1)^k s^k \partial^k \mu (= s^{1-\lambda}\mu_k)$ is a positive measure with the property that

$$\int_0^\infty e^{-xs} s^{\lambda-1} d((-1)^k s^k \partial^k \mu)(s) < \infty.$$

Thus (b) follows. \square

The next result is a special case of (b) implies (a) in Theorem 1.6. We state and prove it separately in order to describe the method, which will be alluded to in the following proof.

Proposition 2.3 *Let f have the representation*

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1} d\mu(s),$$

where $c \geq 0$ and μ is a positive measure on $(0, \infty)$. If $-s\partial\mu(s)$ is a positive measure then

$$c_1^\lambda(f)(x) = \lambda c + \int_0^\infty e^{-xs} s^{\lambda-1} d(-s\partial\mu(s))$$

is completely monotonic.

Proof. Let $n \geq 1$ and take $\varphi_n \in C^\infty((0, \infty))$ such that $0 \leq \varphi_n(t) \leq 1$,

$$\varphi_n(t) = \begin{cases} 0, & t < 1/(2n) \\ 1, & 1/n \leq t \leq n \\ 0, & t > 2n \end{cases},$$

$|\varphi_n'(t)| \leq \text{Const} \cdot n$ for $t \in (1/(2n), 1/n)$, and $|\varphi_n'(t)| \leq \text{Const}$ for $t \in (n, 2n)$.

By definition of the derivative in distributional sense we have

$$\begin{aligned} & \int_0^\infty e^{-xs} s^{\lambda-1} \varphi_n(s) d(-s\partial\mu(s)) \\ &= \langle -s\partial\mu(s), e^{-xs} s^{\lambda-1} \varphi_n(s) \rangle \\ &= \langle \mu, (e^{-xs} s^\lambda \varphi_n(s))' \rangle \\ &= -x \int_0^\infty e^{-xs} s^\lambda \varphi_n(s) d\mu(s) + \int_0^\infty e^{-xs} \lambda s^{\lambda-1} \varphi_n(s) d\mu(s) \\ & \quad + \int_0^\infty e^{-xs} s^\lambda \varphi_n'(s) d\mu(s). \end{aligned}$$

Using dominated convergence it follows that the sum of the first and second term on the right hand side tends to

$$-x \int_0^\infty e^{-xs} s^\lambda d\mu(s) + \lambda \int_0^\infty e^{-xs} s^{\lambda-1} d\mu(s).$$

The third term tends to zero, again due to dominated convergence and the estimate (using $|s\varphi'(s)| \leq \text{Const}$ for $s \leq 1/n$)

$$\begin{aligned} & \int_0^\infty |e^{-xs} s^\lambda \varphi_n'(s)| d\mu(s) \leq \\ & \text{Const} \left(\int_{1/2n}^{1/n} e^{-xs} s^{\lambda-1} d\mu(s) + \int_n^{2n} e^{-xs} s^\lambda d\mu(s) \right). \end{aligned}$$

Hence, letting n tend to infinity, we obtain that

$$\begin{aligned} c_1^\lambda(f)(x) &= \lambda f(x) + x f'(x) \\ &= \lambda c + \lambda \int_0^\infty e^{-xs} s^{\lambda-1} d\mu(s) - x \int_0^\infty e^{-xs} s^\lambda d\mu(s) \\ &= \lambda c + \int_0^\infty e^{-xs} s^{\lambda-1} d(-s\partial\mu(s)). \end{aligned}$$

Thus $c_1^\lambda(f)$ is completely monotonic, and $e^{-xs} s^{\lambda-1}$ is integrable w.r.t. the measure $-s\partial\mu(s)$. \square

Proof that (b) implies (a) in Theorem 1.6. We suppose that f has the representation

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda-1} d\mu(s),$$

with $c \geq 0$, and $\mu_k \equiv (-1)^k s^k \partial^k \mu$ being a positive measure for $k = 0, \dots, N$. It is easy to verify that $\mu_{k+1} = k\mu_k - s\partial\mu_k$ for $k = 0, \dots, N-1$. Proposition 2.3 yields that $c_1^\lambda(f)$ is completely monotonic and has the representation

$$c_1^\lambda(f)(x) = \lambda c + \int_0^\infty e^{-xs} s^{\lambda-1} d\mu_1(s).$$

Let us now assume that $c_k^\lambda(f)$ is completely monotonic and has the representation

$$c_k^\lambda(f)(x) = b_k + \int_0^\infty e^{-xs} s^{\lambda-1} d\mu_k(s).$$

Then

$$\begin{aligned} c_{k+1}^\lambda(f)(x) &= (\lambda + k) c_k^\lambda(f)(x) + x c_k^\lambda(f)'(x) \\ &= (\lambda + k) \int_0^\infty e^{-xs} s^{\lambda-1} d\mu_k(s) - x \int_0^\infty e^{-xs} s^\lambda d\mu_k(s) \\ &\quad + (\lambda + k) b_k. \end{aligned}$$

Now, taking φ_n as before it follows that

$$\begin{aligned}
& \int_0^\infty e^{-xs} s^{\lambda-1} \varphi_n(s) d\mu_{k+1}(s) \\
&= \int_0^\infty e^{-xs} s^{\lambda-1} \varphi_n(s) d(k\mu_k - s\partial\mu_k)(s) \\
&= \langle k\mu_k - s\partial\mu_k(s), e^{-xs} s^{\lambda-1} \varphi_n(s) \rangle \\
&= \langle k\mu_k, e^{-xs} s^{\lambda-1} \varphi_n(s) \rangle + \langle \mu_k, (e^{-xs} s^\lambda \varphi_n(s))' \rangle \\
&= -x \int_0^\infty e^{-xs} s^\lambda \varphi_n(s) d\mu_k(s) + (k + \lambda) \int_0^\infty e^{-xs} s^{\lambda-1} \varphi_n(s) d\mu_k(s) \\
&\quad + \int_0^\infty e^{-xs} s^\lambda \varphi_n'(s) d\mu_k(s).
\end{aligned}$$

As before, letting n tend to infinity, and applying dominated convergence we get that

$$c_{k+1}^\lambda(f)(x) = \int_0^\infty e^{-xs} s^{\lambda-1} d\mu_{k+1}(s) + (\lambda + k)b_k$$

is completely monotonic.

In the affirmative case we infer that $b_{k+1} = (\lambda + k)b_k$, \dots , $b_1 = \lambda c$ yielding $b_k = (\lambda)_k c$. \square

3 Additional results and comments

Suppose that $c_k^\lambda(f)$ is completely monotonic for some $k \geq 1$. What can be said about the functions $c_0^\lambda(f), \dots, c_{k-1}^\lambda(f)$? Are they also completely monotonic? The answer is given in Proposition 3.1.

Proposition 3.1 *Let $k \geq 1$, $f \in C^\infty((0, \infty))$ and suppose that the functions $c_0^\lambda(f), \dots, c_{k-1}^\lambda(f)$ are non-negative. If $c_k^\lambda(f)$ is completely monotonic then $c_j^\lambda(f)$ is also completely monotonic for $j \leq k - 1$.*

The proof of this proposition requires some preliminary results. Define

$$\gamma_k^\lambda(f)(x) \equiv x^{-\lambda} (x^{\lambda-1+k} f(x))^{(k-1)}, \quad k \geq 1.$$

Notice that $\gamma_k^\lambda(f) = c_{k-1}^{\lambda+1}(f)$.

Lemma 3.2 *For $f \in C^{(k+1)}((0, \infty))$ we have*

$$\gamma_k^\lambda(f)(x) = c_{k-1}^\lambda(f)(x) + (k-1)\gamma_{k-1}^\lambda(f)(x).$$

Proof: This follows by a direct computation. \square

The next lemma is an immediate consequence of Lemma 3.2.

Lemma 3.3 *If $c_j^\lambda(f)(x) \geq 0$ for all $j = 0, \dots, k$ then $\gamma_j^\lambda(f)(x) \geq 0$ for all $j = 1, \dots, k+1$*

Lemma 3.4 *Let $k \geq 1$ be given and assume that $c_j^\lambda(f)(x) \geq 0$ for all $j = 0, \dots, k$. Then:*

- (i) $(x^{\lambda-1+k}f(x))^{(j)} \geq 0$ for all $j = 0, \dots, k$;
- (ii) $\lim_{x \rightarrow 0}(x^{\lambda-1+k}f(x))^{(j)} = 0$ for all $j = 0, \dots, k-2$;
- (iii) $\lim_{x \rightarrow 0}(x^{\lambda-1+k}f(x))^{(k-1)} \in [0, \infty)$;

Proof. We use induction in k . For $k = 1$ (i) is clearly satisfied, (ii) needs not be checked, and (iii) follows by noticing that $x^\lambda f(x)$ is non-negative and increasing. For $k = 2$, $(x^{\lambda+1}f(x))'' = x^{\lambda-1}c_2^\lambda(f)(x) \geq 0$, $(x^{\lambda+1}f(x))' = x^\lambda c_1^\lambda(f)(x) + \lambda x^\lambda c_0^\lambda(f)(x) \geq 0$, and thus (i) is satisfied. Property (ii) is clearly satisfied, and (iii) follows since $(x^{\lambda+1}f(x))'$ is non-negative and increasing.

Next we assume that f satisfies $c_j^\lambda(f) \geq 0$ for all $j \leq k+1$, and aim at verifying (i), (ii), and (iii) with k replaced by $k+1$. For $j = k+1$ we get $(x^{\lambda+k}f(x))^{(j)} = x^{\lambda-1}c_{k+1}^\lambda(f)(x) \geq 0$. For $1 \leq j \leq k$ we use

$$(x^{\lambda+k}f(x))^{(j)} = x(x^{\lambda-1+k}f(x))^{(j)} + j(x^{\lambda-1+k}f(x))^{(j-1)} \geq 0,$$

and (i) is verified. To see (ii), notice that

$$(x^{\lambda+k}f(x))^{(k-1)} = x(x^{\lambda-1+k}f(x))^{(k-1)} + (k-1)(x^{\lambda-1+k}f(x))^{(k-2)},$$

The last term tends to zero by the induction hypothesis, and the first term equals x times a non-negative and increasing function. Hence (ii) holds for $k+1$. Property (iii) for $k+1$ follows since $(x^{\lambda+k}f(x))^{(k)}$ is a positive and increasing function. This proves the lemma. \square

Lemma 3.5 *Let $f \in C^\infty((0, \infty))$ and suppose that $c_0^\lambda(f), \dots, c_{k-1}^\lambda(f)$ are non-negative functions. If $c_k^\lambda(f)$ is completely monotonic then $\gamma_k^\lambda(f)$ is also completely monotonic and*

$$\gamma_k^\lambda(f)(x) = \frac{l_k}{x^\lambda} + \frac{b_k}{\lambda} + \int_0^\infty M_k(u)u^{\lambda-1}e^{-xu} du,$$

where

$$M_k(u) = \int_u^\infty s^{-\lambda} d\mu_k(s).$$

Proof. By the complete monotonicity we may write

$$(x^{\lambda-1+k}f(x))^{(k)} = x^{\lambda-1}c_k^\lambda(f)(x) = x^{\lambda-1} \int_0^\infty e^{-xs} d\mu_k(s) + b_k x^{\lambda-1},$$

where $b_k \geq 0$ and μ_k is a positive measure on $(0, \infty)$. The assumptions on non-negativity yield that the function $x^\lambda \gamma_k^\lambda(f)(x) = (x^{\lambda-1+k}f(x))^{(k-1)}$ is non-negative and increasing. Hence

$$l_k \equiv \lim_{x \rightarrow 0} x^\lambda \gamma_k^\lambda(f)(x) \geq 0.$$

Furthermore,

$$\begin{aligned} x^\lambda \gamma_k^\lambda(f)(x) - l_k &= \int_0^x (t^{\lambda-1+k}f(t))^{(k)} dt \\ &= \int_0^x t^{\lambda-1} \left(\int_0^\infty e^{-ts} d\mu_k(s) + b_k \right) dt \\ &= \frac{b_k}{\lambda} x^\lambda + \int_0^\infty \int_0^x t^{\lambda-1} e^{-ts} dt d\mu_k(s) \\ &= \frac{b_k}{\lambda} x^\lambda + x^\lambda \int_0^\infty \int_0^s u^{\lambda-1} e^{-xu} du s^{-\lambda} d\mu_k(s) \\ &= \frac{b_k}{\lambda} x^\lambda + x^\lambda \int_0^\infty \int_u^\infty s^{-\lambda} d\mu_k(s) u^{\lambda-1} e^{-xu} du, \end{aligned}$$

by Tonelli's theorem. Consequently,

$$M_k(u) = \int_u^\infty s^{-\lambda} d\mu_k(s)$$

is finite and $M_k(u)u^{\lambda-1}$ is integrable at 0.

The formulas above also show that

$$\gamma_k^\lambda(f)(x) = \frac{l_k}{x^\lambda} + \frac{b_k}{\lambda} + \int_0^\infty M_k(u) u^{\lambda-1} e^{-xu} du$$

is completely monotonic. □

Proof of Proposition 3.1: From Lemma 3.5,

$$(x^{\lambda-1+k}f(x))^{(k-1)} = l_k + \frac{b_k}{\lambda} x^\lambda + x^\lambda \int_0^\infty M_k(u) u^{\lambda-1} e^{-xu} du,$$

where $l_k, b_k \geq 0$ and $M_k(u) = \int_u^\infty s^{-\lambda} d\mu_k(s)$. Notice that

$$u^\lambda M_k(u) \rightarrow 0, \quad u \rightarrow 0. \tag{4}$$

(To see this, rewrite as follows

$$u^\lambda M_k(u) = \int_u^1 \left(\frac{u}{s}\right)^\lambda d\mu_k(s) + u^\lambda \int_1^\infty \frac{d\mu_k(s)}{s^\lambda},$$

and use the dominated convergence theorem on the first term.)

Integrating this relation from ϵ to x , and letting ϵ tend to 0 we get, using (ii) of Lemma 3.4, that

$$\begin{aligned} (x^{\lambda-1+k} f(x))^{(k-2)} &= l_k x + \frac{b_k}{\lambda(\lambda+1)} x^{\lambda+1} \\ &\quad + x^{\lambda+1} \int_0^\infty M_{k,k-1}(u) u^\lambda e^{-xu} du, \end{aligned}$$

where

$$M_{k,k-1}(u) = \int_u^\infty \frac{M_k(s)}{s^2} ds.$$

Continuing this process (using in each step (ii) of Lemma 3.4) we get

$$\begin{aligned} x^{\lambda-1+k} f(x) &= \frac{l_k}{(k-1)!} x^{k-1} + \frac{b_k}{(\lambda)_k} x^{\lambda+k-1} \\ &\quad + x^{\lambda+k-1} \int_0^\infty M_{k,1}(u) u^{\lambda+k-2} e^{-xu} du, \end{aligned} \quad (5)$$

where

$$M_{k,k}(u) = M_k(u), \quad M_{k,j}(u) = \int_u^\infty \frac{M_{k,j+1}(s)}{s^2} ds, \quad j = 1, \dots, k-1.$$

To ease notation we write $M_j(u) = M_{k,j}(u)$. (We remark that M_j does not have the same meaning here as in Lemma 3.5.) Division by $x^{\lambda-1+k}$ in (5) shows that f is completely monotonic, and has the representation

$$\begin{aligned} f(x) &= \frac{l_k}{(k-1)!} x^{-\lambda} + \frac{b_k}{(\lambda)_k} + \int_0^\infty M_1(u) u^{\lambda+k-2} e^{-xu} du \\ &= \frac{b_k}{(\lambda)_k} + \int_0^\infty \left(M_1(u) u^{k-1} + \frac{l_k}{(k-1)! \Gamma(\lambda)} \right) e^{-xu} u^{\lambda-1} du. \end{aligned} \quad (6)$$

In order to show that the functions $c_1^\lambda(f), \dots, c_{k-1}^\lambda(f)$ are completely monotonic it suffices (Theorem 1.6) to verify that $(-1)^j \partial^j (M_1(u) u^{k-1}) \geq 0$ for $j = 1, \dots, k-1$. Now,

$$M_1(u) u^{k-1} = u^{k-1} \int_u^\infty \frac{M_2(s)}{s^2} ds = \int_1^\infty \frac{M_2(ut) (ut)^{k-2}}{t^k} dt,$$

so it is enough to verify that $(-1)^j \partial^j (M_2(u)u^{k-2}) \geq 0$ in order to obtain that $(-1)^j \partial^j (M_1(u)u^{k-1}) \geq 0$. Repeating this argument we end up having to verify that

$$(-1)^j \partial^j (M_{k-j+1}(u)u^{j-1}) \geq 0, \quad 1 \leq j \leq k-1. \quad (7)$$

These inequalities are verified using induction. For $j = 1$ it reads $\partial M_k(u) \leq 0$ which is true since M_k is a decreasing function. Next assuming that (7) holds for some $j \leq k-2$ we aim at verifying it for $j+1$. We rewrite the expression $(-1)^{j+1} \partial^{j+1} (M_{k-j}(u)u^{j+1})$ in two ways:

$$\begin{aligned} (-1)^{j+1} \partial^{j+1} (M_{k-j}(u)u^{j+1}) &= (-1)^{j+1} u \partial^{j+1} (M_{k-j}(u)u^j) \\ &\quad + (-1)^{j+1} (j+1) \partial^j (M_{k-j}(u)u^j); \\ (-1)^{j+1} \partial^{j+1} (M_{k-j}(u)u^{j+1}) &= (-1)^{j+1} \partial^j \left(-\frac{M_{k-j+1}(u)}{u^2} u^{j+1} \right) \\ &\quad + (-1)^{j+1} (j+1) \partial^j (M_{k-j}(u)u^j). \end{aligned}$$

Comparing these two identities we infer that

$$(-1)^{j+1} u \partial^{j+1} (M_{k-j}(u)u^j) = (-1)^j \partial^j (M_{k-j+1}(u)u^{j-1}),$$

and thus (7) holds for $j+1$. \square

Remark 3.6 *Introducing the functions $N_j(u) \equiv M_j(1/u)$ for $j = 1, \dots, k$ where M_1, \dots, M_k are those considered in the proof of Proposition 3.1 it follows that*

$$N_k(u) = \int_{1/u}^{\infty} s^{-\lambda} d\mu_k(s) = \int_0^u t^\lambda d\widehat{\mu}_k(t),$$

where $\widehat{\mu}_k$ denotes the image measure $\phi(\mu_k)$, with $\phi(x) = 1/x$. For $j \leq k-1$ the relation between N_j and N_{j+1} is

$$N_j(u) = \int_{1/u}^{\infty} \frac{M_{j+1}(s)}{s^2} ds = \int_0^u N_{j+1}(t) dt.$$

Consequently we see that the derivatives $N_1^{(j)}(u)$ for $j \leq k-1$ are all non-negative, and take the value 0 at $u = 0$. In terms of these functions the representation (6) can be rewritten as

$$\begin{aligned} f(x) &= \frac{b_k}{(\lambda)_k} + \frac{l_k}{(k-1)!x^\lambda} + \int_0^\infty M_1(u)u^{k-1}e^{-xu}u^{\lambda-1} du \\ &= \frac{b_k}{(\lambda)_k} + \frac{l_k}{(k-1)!x^\lambda} + \int_0^\infty N_1(s)s^{-k-\lambda}e^{-x/s} ds. \end{aligned}$$

The next proposition shows that for any given N the classes \mathcal{C}_N^λ become larger as λ increases. As remarked in [9] it is not clear how to verify this even for $N = \infty$ only considering the operators $T_{n,k}^\lambda$.

Proposition 3.7 *If $\lambda_1 < \lambda_2$ then $\mathcal{C}_N^{\lambda_1} \subset \mathcal{C}_N^{\lambda_2}$ for all N .*

Proof. This follows from Leibniz' formula. Assume $f \in \mathcal{C}_N^{\lambda_1}$ and let $\lambda_2 > \lambda_1$. Then

$$f(x) = c + \int_0^\infty e^{-xs} s^{\lambda_1-1} d\mu(s) = c + \int_0^\infty e^{-xs} s^{\lambda_2-1} s^{\lambda_1-\lambda_2} d\mu(s),$$

where $(-1)^j s^j \partial^j \mu \geq 0$ for all $j \leq N$. Hence, for $k \leq N$,

$$\begin{aligned} & (-1)^k s^k \partial^k (s^{\lambda_1-\lambda_2} \mu) \\ &= (-1)^k s^k \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\Gamma(k-j+\lambda_2-\lambda_1)}{\Gamma(\lambda_2-\lambda_1)} s^{\lambda_1-\lambda_2+j-k} \partial^j \mu \\ &= s^{\lambda_1-\lambda_2} \sum_{j=0}^k \binom{k}{j} \frac{\Gamma(k-j+\lambda_2-\lambda_1)}{\Gamma(\lambda_2-\lambda_1)} (-1)^j s^j \partial^j \mu \geq 0. \end{aligned}$$

□

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